Invariant measures and stability of Markov operators

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Definitions

Let (X, ρ) be a Polish space. Let $\mathcal{B}(X)$ be the space of all Borel subsets of X and let $B_b(X)$ (resp. $C_b(X)$) be the Banach space of all bounded, measurable (resp. continuous) functions on X equipped with the supremum norm $\|\cdot\|_\infty$. We denote by $\mathrm{Lip}_b(X)$ the space of all bounded Lipschitz continuous functions on X. By $\mathcal{M}$ and $\mathcal{M}_1$ we denote the family of Borel measures such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$.

An operator $P_* : \mathcal{M} \to \mathcal{M}$ will be called a *Markov operator* if it satisfies the following two conditions

- positive linearity: $P_*(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P_*\mu_1 + \lambda_2 P_*\mu_2$ for $\lambda_1, \lambda_2 \geq 0$; $\mu_1, \mu_2 \in \mathcal{M}$;
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Definitions

A Markov operator P_* is called a *Feller operator* if there is a linear operator $P : C_b(X) \to C_b(X)$ such that

$$\int_X Pf(x)\mu(\mathrm{d}x) = \int_X f(x)P_*\mu(\mathrm{d}x)$$

for any $f \in C_b(X)$ and $\mu \in \mathcal{M}$.
Let P_* be a Markov operator; a measure $\mu \in \mathcal{M}$ is called *invariant* if $P_*\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that

$$\operatorname*{w-lim}_{n\to\infty} P_*^n\mu = \mu_*$$

for any $\mu \in \mathcal{M}_1$.

Invariant measures for Markov operators

Theorem 1. A. Lasota, J. Yorke, Random Comput. Dynam. (1994)

Let $P_* : \mathcal{M} \to \mathcal{M}$ be a Feller operator. Assume that there is a compact set $Y \subset X$ and a measure $\mu_0 \in \mathcal{M}_1$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_*^k \mu_0(Y) > 0.$$

Then there exists an invariant measure $\mu_* \in \mathcal{M}_1$.

Wasserstein metric

In the space $\mathcal{M}$ we introduce the **Wasserstein distance**

$$d_w(\mu,\nu) = \sup\left\{\left|\int_X f\,\mathrm{d}(\mu-\nu)\right| : \|f\|_\infty \le 1, \mathrm{Lip}\, f \le 1\right\}$$

for $\mu,\nu \in \mathcal{M}$.
A Markov operator will be called *nonexpansive* if

$$d_w(P_*\mu, P_*\nu) \le d_w(\mu,\nu) \qquad \text{for } \mu,\nu \in \mathcal{M}_1.$$

Stability of Markov operators

Theorem 2. A. Lasota, J. Yorke, Random Comput. Dynam. (1994)

Let $P_* : \mathcal{M} \to \mathcal{M}$ be a nonexpansive Markov operator. Assume that for every $\varepsilon > 0$ there is a Borel set A with $\operatorname{diam} A \leq \varepsilon$ and a number $\alpha > 0$ such that

$$\liminf_{n\to\infty} P_*^n \mu(A) \geq \alpha \qquad \text{for } \mu \in \mathcal{M}_1.$$

Then P_* is asymptotically stable.

Markov semigroups

Let $((Z(t))_{t\geq 0}$ be a Markov process taking values in X and let $(P^t)_{t\geq 0}$ be its transition semigroup.

We shall assume that the semigroup $(P^t)_{t\geq 0}$ is *Feller*, i.e. $P^t(C_b(X)) \subset C_b(X)$ and that the Markov family is *stochastically continuous*, which implies that: $\lim_{t\to 0+} P_t\psi(x) = \psi(x)$ for all $x \in X$ and $\psi \in C_b(X)$.

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We say that a transition semigroup $(P^t)_{t\geq 0}$ has the *e-property* at $x \in X$ if the family of functions $(P^t\psi)_{t\geq 0}$ is equicontinuous at x for any bounded and Lipschitz continuous function ψ.

The semigroup $(P^t)_{t\geq 0}$ has the e-property if the above condition holds at any $x \in X$.

Let $(P^t_*)_{t\geq 0}$ be the dual semigroup defined on the space $\mathcal{M}_1$ given by the formula

$$P^t_*\mu(B) := \int_X P^t \mathbf{1}_B \mathrm{d}\mu \quad \text{for } B \in \mathcal{B}(X).$$

Recall that $\mu_* \in \mathcal{M}_1$ is *invariant* for the semigroup $(P^t)_{t\geq 0}$ (or the Markov family $(Z(t))_{t\geq 0}$ if $P^t_*\mu_* = \mu_*$ for all $t \geq 0$.

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The set $\mathcal{T}$

For a given $T > 0$ and $\mu \in \mathcal{M}_1$ define

$$Q^T\mu := T^{-1}\int_0^T P_*^s\mu \mathrm{d}s.$$

We write $Q^T(x)$ in the particular case when $\mu = \delta_x$.
The crucial role is played by the set

$$\mathcal{T} := \left\{x \in X : \text{the family of measures } (Q^t(x))_{t\geq 0} \text{ is tight}\right\}.$$

- if $\mathcal{T} \neq \emptyset$, then the semigroup $(P^t)_{t\geq 0}$ admits an invariant measure;
- if μ_* is an invariant measure, then supp $\mu_* \subset \mathcal{T}$;
- if If $x \in \mathcal{T}$, then the sequence $(Q^t(x))_{t\geq 0}$ weakly converges to some invariant measure.

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Theorem 3. (A. Lasota and T.S., J. Diff. Eqs 2006)

Let $(P^t)_{t\geq 0}$ be a Feller semigroup. Assume that there exists a point $z \in X$ such that for every $\delta > 0$

$$\limsup_{T\to\infty} Q^T(x, B(z,\delta)) > 0 \qquad \text{for some } x \in X.$$

If the semigroup $(P^t)_{t\geq 0}$ has the e-property in $z \in X$, then $z \in \mathcal{T}$. Consequently, $(P^t)_{t\geq 0}$ admits an invariant measure.

Invariant measures for Markov semigroups

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Uniqueness of an invariant measure for Markov semigroups

Theorem 4. (T. Komorowski, S. Peszat and T.S., Ann. Prob. 2010)

Assume that $(P^t)_{t\geq 0}$ has the e–property and that there exists a point $z \in X$ such that for every $\delta > 0$ and every $x \in X$

$$\limsup_{T\to\infty} Q^T(x, B(z,\delta)) > 0.$$

Then $(P^t)_{t\geq 0}$ admits a unique invariant measure μ_*. Moreover,

$$\text{w-}\lim_{t\to\infty} Q^t\nu = \mu_*$$

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Ergodic measures

An invariant measure $\mu \in \mathcal{M}_1$ is called **ergodic** if every $A \in \mathcal{B}(X)$ such that $P_t \mathbf{1}_A = \mathbf{1}_A$ for $t \geq 0$ satisfies $\mu(A) \in \{0, 1\}$.

We shall assume the following concentrating condition:
$(\mathcal{C})$ There exists a compact set $K \subset X$ such that for any $\varepsilon > 0$ and every $x \in X$

$$\limsup_{T \to +\infty} Q^T(x, K^\varepsilon) > 0,$$

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We may formulate the following result.

Theorem 5. (D. Worm and T.S., ETDS 2012)

If $(P^t)_{t\geq 0}$ satisfies the e-property and $(\mathcal{C})$, then there exists a Borel set $K_0 \subset K \cap \mathcal{T}$ such that

- $x \in \operatorname{supp} \nu^x$ and ν^x is ergodic for all $x \in K_0$,
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Ergodic measures

Now we fix $x_0 \in X$. For $f : X \to \mathbb{R}$ and $\theta > 0$ we define the local Lipschitz constant

$$|f|_{Lip,\theta} := \sup\left\{ \frac{|f(x) - f(y)|}{\rho(x,y)} : x \neq y;\ x, y \in B(x_0, \theta) \right\}.$$

Proposition 6. (D. Worm and T.S., ETDS 2012)

Let $(P^t)_{t\geq 0}$ satisfy the e-property and $(\mathcal{C})$. If there are sequences $t_n > 0$ and $\delta_n \downarrow 0$ and a non-decreasing function $C : \mathbb{R}_+ \to \mathbb{R}_+$, such that for all bounded and Lipschitz functions and $\theta > 0$

$$|P^{t_n} f|_{Lip,\theta} \leq C(\theta)(\|f\|_\infty + \delta_n \operatorname{Lip} f).$$

Then $(P^t)_{t\geq 0}$ admits only finitely many ergodic measures.

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Let $(P^t)_{t\geq 0}$ satisfy the e-property and $(\mathcal{C})$. If there are sequences $t_n > 0$ and $\delta_n \downarrow 0$ and a non-decreasing function $C : \mathbb{R}_+ \to \mathbb{R}_+$, such that for all bounded and Lipschitz functions and $\theta > 0$

$$|P^{t_n} f|_{Lip,\theta} \leq C(\theta)(\|f\|_\infty + \delta_n \operatorname{Lip} f).$$

Then $(P^t)_{t\geq 0}$ admits only finitely many ergodic measures.

Ergodic measures

Now we fix $x_0 \in X$. For $f : X \to \mathbb{R}$ and $\theta > 0$ we define the local Lipschitz constant

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Application to Stochastic Equations

We study the Markov process defined by the stochastic evolution equation

$$\mathrm{d}Z(t) = (AZ(t) + F(Z(t)))\,\mathrm{d}t + R\mathrm{d}W(t). \tag{1}$$

- A is the generator of a C_0-semigroup $S = (S(t))_{t\geq 0}$ on some real separable Hilbert space $\mathcal{X}$,
- F maps (not necessarily continuously) $D(F) \subset \mathcal{X}$ into $\mathcal{X}$,
- R is a bounded linear operator from another Hilbert space $\mathcal{H}$ to $\mathcal{X}$, and
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We suppose that for every $x \in \mathcal{X}$ there is a unique mild solution $Z^x = (Z_t^x)_{t\geq 0}$ of (**??**) starting at x, and that (**??**) defines in that way a Markov family. We assume that for any $x \in \mathcal{X}$, the process Z^x is stochastically continuous.
The corresponding transition semigroup is given by

$$P_t\psi(x) = \mathbb{E}\,\psi(Z^x(t)),$$

$\psi \in B_b(\mathcal{X})$, and we assume that it is Feller.
A function $\Phi\colon \mathcal{X} \mapsto [0, +\infty)$ will be called a **Lyapunov function**, if it is measurable and

$$\lim_{\|x\|_{\mathcal{X}}\to\infty} \Phi(x) = \infty.$$

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Applications to SPDE's

We shall assume that the deterministic equation

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = AY(t) + F(Y(t)), \qquad Y(0) = x \qquad (2)$$

defines a continuous semi-dynamical system
$Y^x = (Y^x(t),\ t \geq 0)$.

A set $\mathcal{K} \subset \mathcal{X}$ is called a **global attractor** for Y^x if

1) it is invariant under the semi-dynamical system, i.e. $Y^x(t) \in \mathcal{K}$ for any $x \in \mathcal{K}$ and $t \geq 0$.
2) for any $\varepsilon, R > 0$ there exists T such that $Y^x(t) \in \mathcal{K} + \varepsilon B(0,1)$ for $t \geq T$ and $\|x\|_{\mathcal{X}} \leq R$.

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The family $(Z^x(t))_{t\geq 0}$, $x \in \mathcal{X}$, is **stochastically stable** if for every $\varepsilon,\ R,\ t > 0$

$$\inf_{x\in B(0,R)} \mathbb{P}\left(\|Z^x(t) - Y^x(t)\|_{\mathcal{X}} < \varepsilon\right) > 0.$$

Theorem 5. (T. K., S. P. and T.S. Ann. Prob. 2010)

Assume that:

- there exists a global attractor $\mathcal{K}$ of the semi-dynamical system $(Y^x(t),\ t \geq 0)$ defined by (**??**);
- there exists a certain Lyapunov function Φ such that

$$\sup_{t\geq 0} \mathbb{E}\, \Phi(Z^x(t)) < \infty \qquad \text{for any } x \in \mathcal{X},$$

- the family $(Z^x(t))_{t\geq 0}$, $x \in \mathcal{X}$, is stochastically stable, its transition semigroup has the e-property and

$$\bigcap_{x\in\mathcal{K}} \bigcup_{t\geq 0} \Gamma^t(x) \neq \emptyset, \tag{3}$$

where $\Gamma^t(x) = \operatorname{supp} P_t^* \delta_x$

Applications to SPDE's

Theorem 5. (continuation)

Then, $(Z^x(t))_{t\geq 0}$, $x \in \mathcal{X}$ admits a unique invariant measure μ_*. Moreover, we have

$$\operatorname*{w-lim}_{t\to\infty} Q^t\mu = \mu_*$$

for any $\mu \in \mathcal{M}_1$.

If we additionally assume that the attractor $\mathcal{K}$ is a singleton, then $(P^t)_{t\geq 0}$ is asymptotically stable, i.e.

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The CLT and LIL

If we assume additionally that the Markov semigroup $(P_t)_{t\geq 0}$ corresponding to some Markov process $(Z(t))_{t\geq 0}$ is exponentially convergent, i.e., there exists $\alpha > 0$ such that for any Lipschitz function f and $x \in X$ there exists a constant $C := C(f, x) > 0$ such that

$$|P_t f(x) - \int_X f \mathrm{d}\mu_*| \leq Ce^{-\alpha t},$$

where μ_* is a unique invariant measure for the given semigroup, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ such that $\int_X \varphi \mathrm{d}\mu_* = 0$ we obtain:

The CLT and LIL

The Central Limit Theorem: For $W_x(t) = \int_0^t \varphi(Z^x(s))\mathrm{d}s$ we have

$$\frac{W_x(t)}{\sqrt{t}} \Longrightarrow W, \quad \text{as } t \to +\infty,$$

where W is a random variable with normal distribution $\mathcal{N}(0, D)$ and the convergence is understood in law.

The Law of the Iterated Logarithm:

$$\limsup_{t\to+\infty} \frac{W_x(t)}{\sqrt{2t \log\log t}} = D$$

with probability 1. Of course the above implies that also

$$\liminf_{t\to+\infty} \frac{W_x(t)}{\sqrt{2t \log\log t}} = -D$$

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Model of passive tracer

Consider the Navier–Stokes equations (N.S.E.) on a two dimensional torus $\mathbb{T}$,

$$\begin{aligned}
&\partial_t \vec{u}(t,x) + \vec{u}(t,x)\cdot\nabla_x \vec{u}(t,x) = \Delta_x \vec{u}(t,x) - \nabla_x p(t,x) + \vec{F}(t,x),\\
&\nabla\cdot\vec{u}(t,x) = 0,\\
&\vec{u}(0,x) = \vec{u}_0(x).
\end{aligned} \tag{4}$$

The two dimensional vector field $\vec{u}(t,x)$ and scalar field $p(t,x)$ over $[0,+\infty)\times\mathbb{T}$, are called an Eulerian velocity and pressure, respectively. The forcing $\vec{F}(t,x)$ is assumed to be a Gaussian white noise in t, homogeneous and sufficiently regular in x defined over a certain probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Model of passive tracer

The trajectory of a tracer particle is defined as the solution of the ordinary differential equation (o.d.e.)

$$\frac{dx(t)}{dt} = \vec{u}(t, x(t)), \quad x(0) = x_0, \tag{5}$$

where $x_0 \in \mathbb{R}^2$.
Thanks to well known regularity properties of solutions of N.S.E $\vec{u}(t, x)$ possesses continuous modification in x for any $t > 0$. However, since $\vec{u}(t, x)$ needs not be Lipschitz in x, the equation might not define $x(t)$, $t \geq 0$, as a stochastic process over $(\Omega, \mathcal{F}, \mathbb{P})$, due to possible non-uniqueness of solutions.

Model of passive tracer

Let $x_0 \in \mathbb{R}^2$. By a *solution to* (**??**) we mean any $(\mathcal{F}_t)$-adapted process $x(t)$, $t \geq 0$, with continuous trajectories, such that

$$x(t) = x_0 + \int_0^t \vec{u}(s, x(s))ds, \quad \forall\, t \geq 0, \qquad \mathbb{P}\text{-a.s.} \tag{6}$$

In our approach a crucial role is played by the *Lagrangian process*

$$\vec{\eta}(t, x) := \vec{u}(t, x(t) + x), \quad t \geq 0,\ x \in \mathbb{T}$$

that describes the environment from the vantage point of the moving particle. It turns out that its rotation in x,

$$\omega(t, x) = \mathrm{rot}\, \vec{\eta}(t, x) := \partial_2\eta_1(t, x) - \partial_1\eta_2(t, x), \quad t \geq 0,\ x \in \mathbb{T},$$

satisfies a stochastic partial differential equation (s.p.d.e.) that is similar to the stochastic N.S.E.

Model of passive tracer

The position $x(t)$ of the particle at time t, can be represented as an additive functional of the Lagrangian process, i.e.

$$x(t) = \int_0^t \psi_*(\omega(s))ds,$$